9.16.2010

Today the most mathy thing I did was consider a family of group schemes over the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[q]$. I was concretely working with $SL_2 = \operatorname{Spec} \mathbb{C}[a, b, c, d]/(ad-bc-1) =: \operatorname{Spec} R$. Also working with

$$\begin{aligned} T &= \operatorname{Spec} R/(b, c) \\ B &= \operatorname{Spec} R/c \\ B^- &= \operatorname{Spec} R/b \\ U &= \operatorname{Spec} R/(c, a-1, d-1) \\ U^- &= \operatorname{Spec} R/(b, a-1, d-1) \end{aligned}$$

This is not the most economical way to express these guys; e.g. it's more common to write $T = \operatorname{Spec} \mathbb{C}[a^{\pm}]$. Also $U = \operatorname{Spec} \mathbb{C}[b] = \mathbb{A}^1$.

The multiplication map is the same one for $GL_2 = \operatorname{Spec} \mathbb{C}[GL_2]$ where $\mathbb{C}[GL_2] = \mathbb{C}[a, b, c, d]_{ad-bc}$. It is

$$\begin{aligned} \mathbb{C}[GL_2] &\longrightarrow \mathbb{C}[GL_2] \otimes_{\mathbb{C}} \mathbb{C}[GL_2] = \mathbb{C}[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}, \begin{smallmatrix} x & y \\ z & w \end{smallmatrix}]_{(ab-cd)(xw-yz)} \\ a &\mapsto ax + bz \\ b &\mapsto ay + bw \\ c &\mapsto cx + dz \\ d &\mapsto cy + dw \end{aligned}$$

As a sanity check I verified the isomorphism $T \times U \cong B$. Namely on the level rings the multiplication $T \times U \to SL_2$ is

$$\begin{aligned} R &\longrightarrow \mathbb{C}[a^{\pm}] \otimes_{\mathbb{C}} [y] \\ a &\mapsto a \otimes 1 \\ b &\mapsto a \otimes y \\ c &\mapsto 0 \\ d &\mapsto a^{-1} \end{aligned}$$

Clearly it's a surjective map and the kernel is generated by c. So the image of the multiplication map is defined with closed subscheme of SL_2 given by $\operatorname{Spec} R/c = B$.

The question that brought me to think about the above descriptions is when you have a family of group schemes over the affine line say with the fiber over 0 being some kinda strange group and all other fibers being some nice simple group. More specifically say over $q \neq 0$ the fiber is SL_2 and over 0 the fiber is the group S defined as the fiber product

$$\begin{array}{ccc} S & \longrightarrow & B \times B^- \\ \downarrow & & \downarrow \\ T & \xrightarrow{diag} & T \times T \end{array}$$

I came up with the following notation for this group $(t,b,b') \leftrightarrow \left(\begin{smallmatrix} t & b \\ 0 & t^{-1}\end{smallmatrix}, \begin{smallmatrix} t & 0 \\ b' & t^{-1}\end{smallmatrix}\right)$. Then the group law is

$$(t,b,b') \cdot (s,c,c') = (st, tc + s^{-1}b, sb' + t^{-1}c')$$

on the other hand it seems like when you look at the fiber product above for S and look at it as just rings you get that $S = \operatorname{Spec}\mathbb{C}[a^{\pm}, b, c]$. It's not clear to me there there is any map from or to this ring to R; i.e. what is the relation to SL_2?

hmm I seem to be stuck here; I'd like to describe the degeneration as something like $\operatorname{Spec}\mathbb{C}[a,b,c,d]/(ad - tbc - 1)$ but this wrong behavior for $t \neq 0$

can't see this being algebraic; seems like what I need is a step function; maybe can approximate holomorphically and potentially write something as a direct limit of complex holomorphic group schemes?

on the other hand it seems to be the case that $\operatorname{Spec}\mathbb{C}[a,b,c,d]/(ad-1)$ is the limit as $t \to 0$ of the family of group schemes $\operatorname{Spec}\mathbb{C}[a,b,c,d,t]/(ad-1, bt)$ or could also do ct; in any case this gives the Borel subgroups. Maybe can do something with some factorization theorems for SL_2?

there is a map $S \to B$ given by $(t,b,b') \mapsto (t,b,0)$; this also gives an embedding $B \to S$ the other way.

9.18.2010

Some of this old some of this is new; but none of it was texed before.

From $L\mathbb{C}^\times \cong \mathbb{C}^\times \times \mathbb{Z} \times L\mathbb{C}/\mathbb{C} = GL_1 \times \mathbb{Z} \times \exp(L\mathbb{C}/\mathbb{C})$ arises the idea to study representations of $GL_1 \times \mathbb{Z}$ which can also be written as $GL_1 \times \pi_1(GL_1)$.

Interested in central extensions by $\mathbb{C}^\times$; once you fix an integer $n \in \mathbb{Z}$ which should be thought of as a map $\pi_1(GL_1) \to \hom_{\mathbb{Z}}(\pi_1(GL_1, \mathbb{Z})$. Actually probably best to think $\pi_(GL_1)$ as more like $\hom(\mathbb{C}^\times, GL_1)$ and n as a map $\hom(\mathbb{C}^\times, GL_1) \to \hom(GL_1, \mathbb{C}^\times)$.

get $\mathbb{C}^\times \times GL_1 \times \mathbb{Z}$ with group law

$$(z,u,m) \cdot (z',u',m') = (zz'(u')^{nm}, uu', m+m') \tag{1}$$

Now $GL_1 \times \mathbb{Z}$ acts on $H = L^2(S^1, \mathbb{C})$ as follows. Let $f_k \in H$ be the orthonormal basis $f_k(z) = z^k$ where $z = e^{i\theta}$. Then

$$(u,m).f_k(z) = f_{k+nm}(zu^{-1}) = u^{-k-nm} z^{k+nm}$$

Actually this doesn't give a repsentation (see equation (3) below); only after you consider it as a representation for $\mathbb{C}^\times \times GL_1 \times \mathbb{Z}$ where $\mathbb{C}^\times$ acts by scalars does it work.

Now repeat with GL_1 replaced by a torus $T = (\mathbb{C}^\times)^r$. Let $l\colon \pi := \hom(\mathbb{C}^\times, T) \to \hom(T, \mathbb{C}^\times)$. And consider $\mathbb{C}^\times \times T \times \pi$ with group structure

$$(z, \vec{u}, \vec{m}) \cdot (z', \vec{u}', \vec{m}') = (zz' l(\vec{m}, \vec{u}'), \vec{u}\vec{u}', \vec{m} + \vec{m}') \tag{2}$$

continuing the analogy this should act on $L^2((S^1)^r, \mathbb{C})$. If claims like $L^2((S^1)^r, \mathbb{C}) = \prod_1^r L^2(S^1, \mathbb{C})$ and that an orthornormal basis would then be functions like $f_{\vec{k}}(\vec{z}) = (z_1^{k_1}, \ldots z_r^{k_r})$ then the action should be something like

$$(\vec{u},\vec{m}).f_{\vec{k}}(\vec{z}) = f_{\vec{k}+l(\vec{m})}(\vec{z}\cdot \vec{u^{-1}})$$

with this notation, and using stuff like $l(\vec{m},\vec{u}') = (\vec{u}')^{l(\vec{m})}$ it seems to be the case that

$$(1,\vec{u},\vec{m})(1,\vec{u}',\vec{m}').f_{\vec{k}}(\vec{z}) = (l(\vec{m},\vec{u}'),\vec{u}\vec{u}',\vec{m}+\vec{m}').f_{\vec{k}}(\vec{z}) \tag{3}$$

Now bringing this story back to loop groups, you also want to add another factor of $\mathbb{C}^\times$; in order to keep track of all the $\mathbb{C}^\times$'s I denote this latest one as $\mathbb{C}^\times_e$. So for example in the first case I considered, I want to form the group $\mathbb{C}^\times \times GL_1 \times \mathbb{Z} \times \mathbb{C}^*_e$ but I don't want $\mathbb{C}^*_e$ to commute with $GL_1 \times \mathbb{Z}$. So I need to say, for $w \in \mathbb{C}^\times_e$ what $w(1,z,m)w^{-1}$ is. Whatever it is it should be true that

$$w(1,z,m)w\cdot w^{-1}(1,z',m')w^{-1} = w(z^{nm'},zz',m+m')w^{-1}$$

guessing that the answer looks like $w(1,z,m)w^{-1} = (phase, w^m z, m)$ you can work out that the answer is

$$w(1,z,m)w^{-1} = (w^{\frac{n}{2}m^2}, w^m z, m) \tag{4}$$

recall n is the level which in the more general setting was the map l. Probably a similar conjugation action works for the more general case of $\mathbb{C}^\times \times T \times \pi \times \mathbb{C}^\times_e$.

9.20.2010 Actually I think I figured out the analogue to (4) and I want to write it out in gory detail. But I want a better notation than $\vec{m}$ maybe $m.$ or ‘m. I think I'll use the latter so I can write stuff like ‘u' and ‘u^{-1}. But actually since i'm pretty much always dealing with vectors i'm going to suppress the vector notation and just write u, m. Then re-writing (3) becomes

$$(1,u,m)(1,u',m').f_k(z) = ((u')^{lm}, uu', m+m').f_k(z) \tag{5}$$

I don't really like it but it's easy to tex. Also as before $l(m,u') = (u')^{lm}$. Also I'll write

$$(u^m)^{l(m)} = u^{mlm}$$

In this way l is a map $\hom(\mathbb{C}^\times, T) \times \hom(\mathbb{C}^\times, T) \to \hom(\mathbb{C}^\times, \mathbb{C}^\times) \cong \mathbb{Z}$. Now back to getting a conjugation action of $\mathbb{C}^\times \times T \times \pi \times \mathbb{C}^\times_e$. The guess is that

$$w(1,u,m)w^{-1} = (\phi(m), w^m, m)$$

And the following should be true:

$$w(1,u,m)w^{-1}\cdot w(1,u',m')w^{-1} = w((u')^{lm}, uu', m+m')w^{-1}$$

$$\begin{aligned} LHS &= (\phi(m), w^m u, m)\cdot(\phi(m'), w^{m'}u', m') \\ &= (\phi(m)\phi(m')(w^m u')^{lm}, w^{m+m'}uu', m+m') \\ RHS &= (\phi(m+m')(u')^{lm}, w^{m+m'}uu', m+m') \\ &\Rightarrow \phi(m+m') = \phi(m)\phi(m')w^{mlm'} \end{aligned}$$

From this the key observation is to guess $\phi(m) = w^{mlm}$, more appropriately

$$\phi_w(m) = w^{m\frac{l}{2}m}$$

Just because there's been so much short hand let me unwrap it once:

$$\begin{aligned} w^{m\frac{l}{2}m} &= (w^m)^{\frac{l(m)}{2}} \\ &= (w^{m_1}, \dots, w^{m_r})^{\frac{l(m)}{2}} \\ &= (w^{m_1}, \dots, w^{m_r})^{\frac{1}{2}(l(m)_1, \dots, l(m)_r)} \\ &= w^{\frac{m_1 l(m)_1}{2}} \cdots w^{\frac{m_r l(m)_r}{2}} \end{aligned}$$

where really l is an $r \times r$ matrix and $l(m)$ is a vector and $l(m)_i$ is the ith component.

But in order for this definition of $\phi_w(m)$ to work it needs to be the case that $m'lm = mlm'$, i.e. l needs to be symmetric! In summary we have the following group law on $\mathbb{C}^\times \times T \times \pi \times \mathbb{C}_e^\times$

$$(z, u, m, w) \cdot (z', u', m', w') = (zz' w^{m' \frac{l}{2} m'} (w^{m'} u')^{lm}, uw^{m'} u', m + m', ww')$$

holy crap; looks complicated. Tried to check if what I've said is even associative; I don't trust my work but maybe.